

Detecting non-Markovianity of quantum evolution via spectra of dynamical maps

Dariusz Chruściński¹, Chiara Macchiavello^{2,3}, and Sabrina Maniscalco^{4,5}

¹Institute of Physics, Nicolaus Copernicus University, Faculty of Physics, Astronomy and Informatics,
Grudziądzka 5/7, 87-100 Toruń, Poland

²Quit group, Dipartimento di Fisica, Università di Pavia, via A. Bassi 6, I-27100 Pavia, Italy

³Istituto Nazionale di Fisica Nucleare, Gruppo IV, via A. Bassi 6, I-27100 Pavia, Italy

⁴Department of Physics and Astronomy, University of Turku, 20014 Turku, Finland

⁵Department of Applied Physics, School of Science, Aalto University, P.O. Box 11000, FIN-00076 Aalto, Finland

We provide an analysis on non-Markovian quantum evolution based on the spectral properties of dynamical maps. We introduce the dynamical analog of entanglement witness to detect non-Markovianity and we illustrate its behaviour with several instructive examples. It is shown that for a certain class of dynamical maps the shape of the body of accessible states provides a simple non-Markovianity witness.

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Introduction — Spectral Theorem is one of the mathematical pillars of quantum theory [1]. This celebrated result of von Neumann states that for any normal operator (i.e. $AA^\dagger = A^\dagger A$) in the Hilbert space one has the corresponding spectral decomposition $A = \sum_k a_k |\phi_k\rangle\langle\phi_k|$ with complex a_k and $\langle\phi_k|\phi_l\rangle = \delta_{kl}$. In particular if A is not only normal but also Hermitian then a_k are real. Many problems from quantum physics are directly related to finding the spectrum $\{a_k\}$ of some normal/Hermitian operator.

In this Letter we apply some tools from spectral analysis to study the evolution of an open quantum system. Such systems provide a fundamental tool to study the interaction between a quantum system and its environment, causing dissipation, decay, and decoherence [2–4]. It is, therefore, clear that open quantum systems are important for quantum-enhanced applications, as both entanglement and quantum coherence are basic resources in modern quantum technologies, such as quantum communication, cryptography, and computation [5].

Recently, much effort has been devoted to the description, analysis and classification of non-Markovian quantum evolution (see e.g. recent review papers [6, 7]). In analogy to entanglement theory [8] several non-Markovianity measures were proposed which characterize various concepts of non-Markovianity. The two most influential approaches to non-Markovian evolution are based on divisibility of dynamical maps [9, 11] and distinguishability of states [12] (for other approaches see also [13–18, 20]). The results we present in this Letter allow to introduce for the first time a witness of non-Markovianity in the same spirit of entanglement witnesses. An entanglement witness method applied to the Choi-Jamiolkowski state of a quantum channel was recently developed [21] in order to detect properties based on convexity features. The method was tested experimentally for entanglement breaking channels and for separable random unitary channels [22].

Besides the fundamental interest, our approach simplifies, in certain cases, the experimental detection of non-

Markovianity of a dynamical map.

Let us recall that a dynamical map Λ_t is CP-divisible if for any $t > s$ one has $\Lambda_t = V_{t,s}\Lambda_s$, with $V_{t,s}$ being completely positive. We call quantum evolution Markovian iff the corresponding dynamical map is CP-divisible. Recently, this notion was refined as follows [23]: Λ_t is k -divisible iff $V_{t,s}$ is k -positive. In particular 1-divisible maps are called P-divisible ($V_{t,s}$ is positive). Maps which are even not P-divisible were called *essentially non-Markovian*. These types of dynamical maps have been recently simulated and detected experimentally [24].

Note that CP-divisibility is a mathematical property of the map. Another approach more operationally oriented is based on distinguishability of quantum states [12]: we call quantum evolution BLP-Markovian if

$$\frac{d}{dt} \|\Lambda_t[\rho_1 - \rho_2]\|_1 \leq 0, \quad (1)$$

for any pair of initial states ρ_1 and ρ_2 . Actually, assuming that Λ_t is invertible one shows [23] that Λ_t is k -divisible iff $\frac{d}{dt} \|(\mathbb{1}_k \otimes \Lambda_t)[X]\|_1 \leq 0$, for all Hermitian $X \in M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H})$. Note, that if $k = 1$ and $X = \rho_1 - \rho_2$ one recovers (1).

In the following we develop further the analysis of non-Markovian evolution based on the spectral properties of dynamical maps, and provide the dynamical analog of entanglement witness for detecting non-Markovianity. Our analysis is restricted to a class of commutative dynamical maps. However, the majority of well known examples of open systems dynamics belongs to this class of maps.

Volume and body of accessible states — Let us denote by B the space of density operators. Clearly $B(t) = \Lambda_t[B]$ denotes the body of accessible states at time t . In a recent paper [18] an interesting geometric characterization is proposed, namely, if Λ_t is P-divisible, then

$$\frac{d}{dt} \text{Vol}(t) \leq 0, \quad (2)$$

where $\text{Vol}(t)$ denotes the volume of accessible states at time t , i.e. the volume of the convex body $B(t)$. This result follows from the fact that $\text{Vol}(t) = |\text{Det}\Lambda_t| \text{Vol}(0)$ and for P-divisible map one has $\frac{d}{dt}|\text{Det}\Lambda_t| \leq 0$ (cf. [19]).

Let us provide more geometrical insight passing to the matrix representation $\Lambda_t \rightarrow F_{\alpha\beta}(t) := \text{Tr}(G_\alpha \Lambda_t [G_\beta])$, where G_α is an orthonormal basis in $\mathcal{B}(\mathcal{H})$. A suitable choice of G_α is the set of generalized Gell-Mann matrices with $G_0 = \mathbb{I}/\sqrt{d}$ and Hermitian G_α ($\alpha = 1, \dots, d^2 - 1$). In this case $F(t)$ has the following form

$$F(t) = \left(\begin{array}{c|c} 1 & 0 \\ \hline \mathbf{q}_t & \Delta_t \end{array} \right), \quad (3)$$

with $\mathbf{q}_t \in \mathbb{R}^{d^2-1}$ and Δ_t being $(d^2 - 1) \times (d^2 - 1)$ real matrix. It is clear that $F(t)$ encodes all properties of the original dynamical map Λ_t . In particular Λ_t and $F(t)$ have exactly the same spectrum $\lambda_\alpha(t)$ ($\alpha = 0, 1, \dots, d^2 - 1$, where $d = \dim \mathcal{H}$), and hence $\text{Det}\Lambda_t = \text{Det}F(t) = \text{Det}\Delta_t$. This shows that the volume of the set of accessible states is fully controlled by the matrix Δ_t itself. Now, defining the generalized Bloch representation $\rho = \frac{1}{d}(\mathbb{I} + \sum_{\alpha=1}^{d^2-1} x_\alpha G_\alpha)$, the action of the channel Λ_t on ρ corresponds to the following affine transformation of the generalized Bloch vector $\mathbf{x} \rightarrow \mathbf{x}_t = \Delta_t \mathbf{x} + \mathbf{q}_t$. If \mathbf{x}_1 and \mathbf{x}_2 are Bloch vectors corresponding to ρ_1 and ρ_2 , then $[\rho_1 - \rho_2] \rightarrow \Lambda_t[\rho_1 - \rho_2]$ corresponds to linear transformation $\Delta_t(\mathbf{x}_1 - \mathbf{x}_2)$ and hence does not depend upon the vector \mathbf{q}_t . It clearly shows that BLP-Markovianity is controlled only by Δ_t whereas the full P-divisibility by the entire map $F(t)$, i.e. both Δ_t and \mathbf{q}_t . Note that divisibility of $F(t)$, that is, $F(t) = F(t, s)F(s)$ implies quite nontrivial relations $\Delta_t = \Delta_{t,s}\Delta_s$ and $\mathbf{q}_t = \mathbf{q}_{t,s} + \Delta_{t,s}\mathbf{q}_s$, where $\mathbf{q}_{t,s}$ and $\Delta_{t,s}$ parameterize $F(t, s)$. They considerably simplify if the dynamical map Λ_t is unital. In this case $\mathbf{q}_t = 0$ and one is left with a simple divisibility condition $\Delta_t = \Delta_{t,s}\Delta_s$.

Proposition 1 *If Λ_t is P-divisible and unital, then*

$$\frac{d}{dt} \|\Lambda_t[X]\|_2 \leq 0, \quad (4)$$

for all normal operators X .

For the proof see [10]. In particular $\frac{d}{dt} \|\Delta_t \mathbf{x}\|_2 \leq 0$ which shows that the Euclidean norm of the Bloch vector \mathbf{x} decreases monotonically [10].

It should be clear that the volume of accessible states provides a rather weak witness – one may easily construct maps satisfying (2) which are not P-divisible. In particular it says nothing about the shape of the body of accessible states. For example very often during the evolution of a qubit the initial Bloch ball is deformed to an ellipsoid. P-divisible dynamics always decreases its volume but what about the length of the corresponding axis? Could one relate P-divisible evolution to the shape of accessible states? Moreover, it should be stressed that the shape is controlled by singular values of $F(t)$ and not

by the spectrum itself. To analyze this problem let us consider singular value decomposition of the matrix $F(t)$

$$F(t) = \mathcal{O}_1(t) \Sigma(t) \mathcal{O}_2^{-1}(t), \quad (5)$$

where $\mathcal{O}_k(t)$ ($k = 1, 2$) are orthogonal matrices and $\Sigma(t)$ is a diagonal matrix containing singular values of $F(t)$. Hence the action of $F(t)$ consists in a rotation $\mathcal{O}_2^{-1}(t)$, a contraction governed by $\Sigma(t)$ (all singular values $\sigma_k(t) \leq 1$) followed by the rotation $\mathcal{O}_1(t)$. Since rotation does not change the volume the latter is fully controlled by $\Sigma(t)$. However, the shape of $B(t)$ depends both upon $\mathcal{O}_1(t)$ and $\mathcal{O}_2(t)$.

Commutative maps — To provide a stronger witness we restrict our analysis to a class of quantum evolutions satisfying the following commutativity condition

$$\Lambda_t \Lambda_s = \Lambda_s \Lambda_t, \quad (6)$$

for any $t, s > 0$. Equivalently, the time-local generator satisfies $\mathcal{L}_t \mathcal{L}_s = \mathcal{L}_s \mathcal{L}_t$. Commutativity condition (6) implies that Λ_t and its dual (Heisenberg picture) possess time independent eigenvectors

$$\Lambda_t[X_\alpha] = \lambda_\alpha(t) X_\alpha, \quad \Lambda_t^*[Y_\alpha] = \lambda_\alpha^*(t) Y_\alpha, \quad (7)$$

for $\alpha = 0, 1, \dots, d^2 - 1$. This condition is indeed very restrictive. However, in practice the majority of the examples considered in the literature belong to the commutative class. The reason is very simple: assuming that Λ_t satisfies the time-local master equation $\frac{d}{dt} \Lambda_t = \mathcal{L}_t \Lambda_t$, with suitable time-local generator \mathcal{L}_t , one has $\Lambda_t = \mathcal{T} e^{\int_0^t \mathcal{L}_\tau d\tau}$, where \mathcal{T} denotes the chronological operator. In general the above formula has only a formal meaning and it is defined by the Dyson expansion $\Lambda_t = \mathbb{I} + \int_0^t dt_1 \mathcal{L}_{t_1} + \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{L}_{t_1} \mathcal{L}_{t_2} + \dots$. Now, in the commutative case (6) the chronological product drops out and the solution is represented by the simple exponential formula $\Lambda_t = e^{\int_0^t \mathcal{L}_\tau d\tau}$. Moreover, the eigenvalues $\lambda_\alpha(t)$ of the dynamical map are related to the corresponding eigenvalues $\mu_\alpha(t)$ of the time-local generator \mathcal{L}_t via $\lambda_\alpha(t) = e^{\int_0^t \mu_\alpha(\tau) d\tau}$. One has, therefore, the following obvious property:

Proposition 2 *If Λ_t defines commutative P-divisible map, then*

$$\frac{d}{dt} |\lambda_\alpha(t)| \leq 0, \quad (8)$$

or equivalently $\text{Re} \mu_\alpha(t) \leq 0$ for $\alpha = 1, \dots, d^2 - 1$.

It is clear that the set of inequalities (8) is much more restrictive than the single condition (2) which immediately follows from (8) due to $\text{Det}F(t) = |\lambda_1(t) \dots \lambda_{d^2-1}(t)|$. Let us observe that for commutative maps condition (2) may be easily translated to the condition upon the time-local generator \mathcal{L}_t . Using the well known property of

matrices $\text{Det } e^A = e^{\text{Tr} A}$ one finds that (2) is equivalent to

$$\text{Tr} \mathcal{L}_t \leq 0, \quad (9)$$

where with $\text{Tr} \mathcal{L}_t$ we mean the sum of eigenvalues or equivalently the trace of the matrix $L(t)$ defined by $L_{\alpha\beta}(t) = \text{Tr}(G_\alpha \mathcal{L}_t[G_\beta])$. In entanglement theory one defines an entanglement witness, i.e. an Hermitian operator W in $\mathcal{H} \otimes \mathcal{H}$ such that: *i)* $\langle \psi_1 \otimes \psi_1 | W | \psi_1 \otimes \psi_1 \rangle \geq 0$, and *ii)* $\text{Tr}(W\rho) < 0$ for some entangled state ρ . Any such operator may be constructed as $W := (\mathbb{1} \otimes \Phi)|\alpha\rangle\langle\alpha|$, where $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a positive but not completely positive map, and

$$|\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i \otimes i\rangle, \quad (10)$$

denotes the maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$. Consider now an arbitrary linear map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ and define

$$f_\Phi = \langle \alpha | (\mathbb{1} \otimes \Phi)[P^+] | \alpha \rangle, \quad (11)$$

with $P^+ = |\alpha\rangle\langle\alpha|$. Interestingly, f_Φ is fully characterized by the spectral properties of the map Φ . One has the following

Proposition 3 *Function f_Φ is fully determined by the spectrum of Φ , that is, $f_\Phi = d^{-2} \sum_{\alpha=0}^{d^2-1} \lambda_\alpha$, where λ_α are eigenvalues of Φ .*

Indeed, consider the spectral representation $\Phi[\rho] = \sum_\alpha \lambda_\alpha F_\alpha \text{Tr}(G_\alpha^\dagger \rho)$, where $\{F_\alpha, G_\alpha\}$ provide a damping basis [26] for the map Φ , that is, $\Phi[F_\alpha] = \lambda_\alpha F_\alpha$ and $\Phi^*[G_\alpha] = \lambda_\alpha^* G_\alpha$ such that $\text{Tr}(F_\alpha G_\beta^\dagger) = \delta_{\alpha\beta}$. One has

$$\begin{aligned} d^2 f_\Phi &= \sum_{i,j} \sum_{k,l} \text{Tr}(|i\rangle\langle j| \otimes |i\rangle\langle j| \cdot (|k\rangle\langle l| \otimes \Phi[|k\rangle\langle l|])) \\ &= \sum_\alpha \sum_{i,j} \lambda_\alpha \langle i | F_\alpha | j \rangle \langle j | G_\alpha^\dagger | i \rangle = \sum_\alpha \lambda_\alpha, \end{aligned}$$

due to $\text{Tr}(F_\alpha G_\alpha^\dagger) = 1$. If the corresponding dynamical map $\Lambda_t = \exp(\int_0^t \mathcal{L}_\tau d\tau)$ is commutative then (2) is equivalent to

$$\langle \alpha | (\mathbb{1} \otimes \mathcal{L}_t)[P^+] | \alpha \rangle \leq 0. \quad (12)$$

Hence, the violation of (12) may be considered as *dynamical analog of an entanglement witness*.

Normal commutative maps — Consider a class of commutative maps (6) which are normal

$$\Lambda_t \Lambda_t^* = \Lambda_t^* \Lambda_t, \quad (13)$$

for any $t \geq 0$. This condition guarantees that due to the spectral theorem both Λ_t and its dual Λ_t^* (Heisenberg picture) possess common eigenvectors, i.e. $X_\alpha = Y_\alpha$.

Moreover, since Λ_t^* is unital, i.e. $\Lambda_t^*[\mathbb{I}] = \mathbb{I}$, the map Λ_t is unital as well. One may choose therefore $X_0 = \mathbb{I}/\sqrt{d}$, with $d = \dim \mathcal{H}$, which corresponds to $\lambda_0(t) = 1$. Using the matrix representation (3) it means that $\mathbf{q}_t = 0$ and Δ_t is a normal matrix. It is well known that the shape of the body of states is not controlled by eigenvalues $\lambda_\alpha(t)$ but by singular values $s_\alpha(t)$. Note, however, that if the map is normal, then $s_\alpha(t) = |\lambda_\alpha(t)|$ and hence conditions (8) are equivalent to $\frac{d}{dt} s_\alpha(t) \leq 0$. Let us observe that the formula (5) implies the following

Proposition 4 *If Λ_t is a P-divisible commutative normal dynamical map, then there exists a family of orthogonal matrices $\mathcal{O}(t, s) \in O(d^2 - 1)$ such that*

$$\mathcal{O}(t, s)[B(t)] \subset B(s), \quad (14)$$

for all $t \geq s$.

The role of $\mathcal{O}(t, s)$ is to rotate $B(t)$ with respect to $B(s)$ such that $\mathcal{O}(t, s)[B(t)]$ is contained within $B(s)$. Hence, for P-divisible commutative normal maps Λ_t , not only $\text{Vol}(t)$ decreases in time but also the body itself $B(t)$ (up to orthogonal rotation) shrinks in time.

Hermitian commutative maps — We conclude our theoretical analysis by considering the most restrictive class of commutative maps, namely those satisfying $\Lambda_t^* = \Lambda_t$. In this case $\lambda_\alpha(t)$ are real and since $\lambda_\alpha(0) = 1$ and the map itself is invertible we have $\lambda_\alpha(t) = |\lambda_\alpha(t)| = s_\alpha(t)$. In this case one has

Proposition 5 *If Λ_t is a P-divisible commutative Hermitian dynamical map, then $B(t) \subset B(s)$, for all $t \geq s$.*

Interestingly, in the case of Hermitian commutative maps we may provide an extra tool for analyzing P-divisibility (or equivalently BLP-Markovianity). For a given dynamical map Λ_t let us define the function

$$f(t) = \langle \alpha | (\mathbb{1} \otimes \Lambda_t)[P^+] | \alpha \rangle = d^{-2} \text{Tr} F(t). \quad (15)$$

One has the following

Proposition 6 *If Λ_t is a P-divisible commutative Hermitian map, then*

$$\frac{d}{dt} f(t) \leq 0, \quad (16)$$

for all $t \geq 0$.

Indeed, since $\lambda_\alpha(0) = 1$, then if $\lambda_\alpha(t)$ is real it must be positive otherwise it would take null value and the generator \mathcal{L}_t would become singular. One has therefore

$$\frac{d}{dt} \sum_\alpha \lambda_\alpha(t) = \frac{d}{dt} \sum_\alpha |\lambda_\alpha(t)| \leq 0,$$

due to (8). Note, that it is enough that Λ_t is commutative and all eigenvalues are real (see Example -

Amplitude damping channel). We note here that condition (16) can be easily detected in an experimental scenario without performing all the measurements required for quantum process tomography. Actually, $f(t)$ is the probability of projecting the global state of the system and the ancilla onto state $|\alpha\rangle$. Let us consider for simplicity the case of two-dimensional systems. We can write $|\alpha\rangle$ in terms of local Pauli operators as $|\alpha\rangle\langle\alpha| = 1/4(I \otimes I + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$. This means that $f(t)$ can be measured from the expectation value of the local observables $\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z$ without requiring a complete set of two-qubit operators that would be needed for entanglement-assisted quantum process tomography. Moreover, the detection scheme considered here would be particularly suited in a linear optical scenario. Actually, the projection onto the maximally entangled state $|\alpha\rangle\langle\alpha|$ could be performed in a single measurement because it corresponds to a single projection onto a Bell state while there is no need to distinguish between the four Bell states, which is usually considered a drawback of linear optical implementations.

Examples of commutative dynamical maps — Interestingly, this restricted class of maps – commutative and normal/Hermitian – still covers many interesting examples.

Example 1 (Qubit dephasing) Consider $\mathcal{L}_t[\rho] = \frac{1}{2}\gamma(t)(\sigma_z \rho \sigma_z - \rho)$, which gives rise to commutative Hermitian dynamical map. For this very simple example all known conditions are equivalent: P -divisibility (equivalently BLP-Markovianity) is equivalent to CP-divisibility, i.e., $\gamma(t) \geq 0$. In this case $B(t)$ defines an axially symmetric ellipsoid $\frac{x_1^2}{\lambda^2(t)} + \frac{x_2^2}{\lambda^2(t)} + x_3^2 \leq 1$, where $\lambda(t) = e^{-\Gamma(t)}$ and $\Gamma(t) = \int_0^t \gamma(u)du$. Hence, this evolution is Markovian iff $B(t) \subset B(s)$ for $t > s$.

This example may be generalized for $d > 2$ in two ways by providing normal or Hermitian time-local generator \mathcal{L}_t . Note that $\{\sigma_0 = \mathbb{I}/\sqrt{2}, \sigma_1/\sqrt{2}, \sigma_2/\sqrt{2}, \sigma_3/\sqrt{2}\}$ define an orthonormal basis in $M_2(\mathbb{C})$ consisting of elements which are both Hermitian and unitary. Now, the unitary basis in $M_d(\mathbb{C})$ is defined by Weyl operators $U_{kl} = \sum_{m=0}^{d-1} \omega^{mk} |m\rangle\langle m+l|$, with $\omega = e^{2\pi i/d}$. If $d = 2$ they reproduce four Pauli matrices. Observe that $U_{k0} = \sum_{m=0}^{d-1} \omega^{mk} |m\rangle\langle m|$ are diagonal and may be used to generalize Example 1.

Example 2 (Qudit dephasing – normal) Consider a qudit generator

$$\mathcal{L}_t[\rho] = \frac{1}{2} \sum_{k=1}^{d-1} \gamma_k(t) (U_{k0} \rho U_{k0}^\dagger - \rho). \quad (17)$$

Clearly for $d = 2$ one has $U_{10} = \sigma_z$ and the above generator reduces to that from Example 1. It is easy to check that \mathcal{L}_t is normal and commutative.

Consider now the Hermitian basis in $M_d(\mathbb{C})$ defined by Gell-Mann matrices. Diagonal elements are defined by

$$V_l = \frac{1}{\sqrt{l(l+1)}} \left(\sum_{k=0}^{l-1} |k\rangle\langle k| - l|l\rangle\langle l| \right), \quad l = 1, \dots, d-1,$$

and for $d = 2$ one has $V_1 = \sigma_z/\sqrt{2}$.

Example 3 (Qudit dephasing – Hermitian)

Consider a qudit generator

$$\mathcal{L}_t[\rho] = -\frac{1}{2} \sum_{l=1}^{d-1} \gamma_l(t) [V_l, [V_l, \rho]]. \quad (18)$$

It is easy to check that \mathcal{L}_t is Hermitian and commutative. Diagonal elements ρ_{kk} do not evolve in time and off-diagonal are multiplied by a function of local decoherence rates $\gamma_k(t)$.

Example 4 (Perfect decoherence – normal)

Consider the following time-independent Hamiltonian in $\mathcal{H}_A \otimes \mathcal{H}_B$

$$H = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B + \sum_k P_k \otimes B_k, \quad (19)$$

where $P_k = |k\rangle\langle k|$ are projectors into the computational basis vectors $|k\rangle$ in \mathcal{H}_A and B_k are hermitian operators in \mathcal{H}_B . Assuming that $H_A = \sum_k \epsilon_k P_k$ one finds $H = \sum_k P_k \otimes Z_k$, where $Z_k = \epsilon_k \mathbb{I}_B + H_B + B_k$. Such Hamiltonian leads to a pure decoherence of the density operator ρ_A of subsystem A:

$$\rho_A(t) = \text{tr}_B(e^{-iHt} \rho_A \otimes \rho_B e^{iHt}) = \sum_{k,l} c_{kl}(t) P_k \rho_A P_l,$$

with $c_{kl}(t) = \text{tr}(e^{-iZ_k t} \rho_B e^{iZ_l t})$. It turns out that $c_{kl}(t)$ define eigenvalues of the map $\Lambda_t[\rho_A] = \sum_{k,l} c_{kl}(t) P_k \rho_A P_l$ which is commutative and normal.

Example 5 (Pauli channel – Hermitian) The qubit dephasing may be immediately generalized to

$$\mathcal{L}_t[\rho] = \frac{1}{2} \sum_{k=1}^3 \gamma_k(t) (\sigma_k \rho \sigma_k - \rho), \quad (20)$$

which lead to the following dynamical map (time-dependent Pauli channel): $\Lambda_t[\rho] = \sum_{\alpha=1}^3 p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha$. It was shown [25] that (8) implies: $\gamma_1(t) + \gamma_2(t) \geq 0$, $\gamma_2(t) + \gamma_3(t) \geq 0$, and $\gamma_3(t) + \gamma_1(t) \geq 0$. In this case $B(t)$ defines an ellipsoid $\frac{x_1^2}{\lambda_1^2(t)} + \frac{x_2^2}{\lambda_2^2(t)} + \frac{x_3^2}{\lambda_3^2(t)} \leq 1$ and this evolution is BLP-Markovian iff $B(t) \subset B(s)$ for $t > s$.

Example 6 (Weyl channel – normal) Pauli channel may be easily generalized for $d > 2$ as follows

$$\mathcal{L}_t[\rho] = \sum_{k+l>0}^{d-1} \gamma_{kl}(t) [U_{kl} \rho U_{kl}^\dagger - \rho], \quad (21)$$

where U_{kl} are Weyl operators. This gives rise to the normal commutative dynamical map $\Lambda_t[\rho] = \sum_{k,l=0}^{d-1} p_{kl}(t) U_{kl} \rho U_{kl}^\dagger$. Conditions (8) lead to $\sum_{k+l>0} \gamma_{kl}(t) [1 - \text{Re} \omega^{mk-nl}] \geq 0$ for all pairs (m, n) .

Example 7 (Generalized Pauli channel)

Generalized Pauli channel [28, 29] is a special example of the Weyl channel defined as follows: let $\{|\psi_0^{(\alpha)}\rangle, \dots, |\psi_{d-1}^{(\alpha)}\rangle\}$ denote $d+1$ mutually unbiased bases (MUBs) in \mathbb{C}^d . Define the quantum channels $\mathcal{P}_\alpha[\rho] = \sum_{l=0}^{d-1} |\psi_l^{(\alpha)}\rangle \langle \psi_l^{(\alpha)}| \rho |\psi_l^{(\alpha)}\rangle \langle \psi_l^{(\alpha)}|$ and let

$$\mathcal{L}_t[\rho] = \sum_{\alpha=1}^{d+1} \gamma_\alpha(t) (\mathcal{P}_\alpha[\rho] - \rho), \quad (22)$$

This map is Hermitian and BLP-Markovianity implies [29] $\gamma(t) - \gamma_\alpha(t) \geq 0$, where $\gamma(t) = \sum_\alpha \gamma_\alpha(t)$.

Example 8 (Amplitude damping channel) The dynamics of a single amplitude-damped qubit is governed by a single function $G(t)$

$$\Lambda_t[\rho] = \begin{pmatrix} \rho_{11} + (1 - |G(t)|^2) \rho_{22} & G(t) \rho_{12} \\ G^*(t) \rho_{21} & |G(t)|^2 \rho_{22} \end{pmatrix}, \quad (23)$$

where the function $G(t)$ depends on the form of the reservoir spectral density $J(\omega)$ [2]. The dynamical map Λ_t is commutative but not normal. The corresponding eigenvalues read as follows: $\lambda_0(t) = 1$, $\lambda_1(t) = G(t)$, $\lambda_2(t) = G^*(t)$, and $\lambda_3(t) = |G(t)|^2$. This evolution is

generated by the following time-local generator

$$\mathcal{L}_t[\rho] = -\frac{is(t)}{2} [\sigma_+ \sigma_-, \rho] + \gamma(t) (\sigma_- \rho_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \rho\}),$$

where σ_\pm are the spin lowering and rising operators together with $s(t) = -2\text{Im} \frac{\dot{G}(t)}{G(t)}$, and $\gamma(t) = -2\text{Re} \frac{\dot{G}(t)}{G(t)}$. It is clear that (8) implies $\gamma(t) \geq 0$. Again in this case this condition is necessary and sufficient for Markovianity. Since eigenvalues are in general complex we cannot use condition (16). Interestingly, for Lorentzian spectral density $J(\omega) = \frac{\gamma_M \lambda^2}{2\pi[(\omega - \omega_c)^2 + \lambda^2]}$ the function $G(t)$ becomes real and hence $f(t) = \frac{1}{4}[1 + G(t)]^2$ and condition (16) implies $\gamma(t) \geq 0$. This example may be considered as an analog of non-Hermitian Hamiltonian with real spectra analyzed by Bender [30].

Conclusions — In this Letter we provided further characterization of non-Markovian evolution for a class of commutative dynamical maps. In this case P-divisibility implies simple conditions for the spectrum of the dynamical map. Moreover, if the map is normal then P-divisibility is equivalent to BLP-Markovianity and the body of accessible states $B(t)$ is contained up to orthogonal rotation in $B(s)$ for $t > s$. This provides a much stronger non-Markovianity witness than the volume of accessible states [18]. Finally, it is argued that the quantity $\langle \alpha | (\mathbb{1} \otimes \mathcal{L}_t)[P^+] | \alpha \rangle$ may be considered as a dynamical analog of entanglement witness that can be easily accessed in the experimental scenario. Our analysis is illustrated by several paradigmatic examples.

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SUPPLEMENTARY MATERIAL

Any positive trace-preserving map is a contraction in the trace norm

$$\|\Phi[X]\|_1 \leq \|X\|_1,$$

where $\|X\|_1 = \text{Tr}|A| = \text{Tr}\sqrt{XX^\dagger}$. Now, if Φ is not only trace-preserving but also unital (doubly stochastic), then it is also a contraction in the Hilbert-Schmidt norm

$$\|\Phi[X]\|_2 \leq \|X\|_2,$$

for all normal X . Recall that $\|X\|_2 = \sqrt{\text{Tr}XX^\dagger}$. Indeed, using the Kadison inequality,

$$\Phi[X^\dagger X] \geq \Phi[X^\dagger]\Phi[X],$$

one has

$$\begin{aligned} \|\Phi[X]\|_2^2 &= \text{Tr}(\Phi[X^\dagger]\Phi[X]) \leq \text{Tr}(\Phi[X^\dagger X]) \\ &= \text{Tr}(X^\dagger X) = \|X\|_2^2. \end{aligned}$$

Suppose now that Λ_t is P-divisible and unital. One has for any normal X

$$\begin{aligned} \frac{d}{dt}\|\Lambda_t[X]\|_2 &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\|\Lambda_{t+\epsilon}[X]\|_2 - \|\Lambda_t[X]\|_2 \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\|V_{t+\epsilon,t}[\Lambda_t[X]]\|_2 - \|\Lambda_t[X]\|_2 \right) \\ &\leq \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\|\Lambda_t[X]\|_2 - \|\Lambda_t[X]\|_2 \right) = 0, \end{aligned}$$

where we used the fact that $V_{t+\epsilon,t}$ is a contraction in Hilbert-Schmidt norm.

To prove

$$\frac{d}{dt}\|\Delta_t \mathbf{x}\|_2 \leq 0,$$

let us consider a Hermitian (and hence normal) operator

$$X = x_0 \mathbb{I} + \sum_k x_k G_k.$$

One has

$$\begin{aligned} \|\Lambda_t[X]\|_2^2 &= \text{Tr} \left[\left(x_0 \mathbb{I} + \sum_k x_k \Lambda_t[G_k] \right) \left(x_0 \mathbb{I} + \sum_l x_l \Lambda_t[G_l] \right) \right] \\ &= x_0^2 d + 2x_0 \sum_k x_k \text{Tr}(\Lambda_t[G_k]) + \sum_{k,l} x_k x_l \text{Tr}(\Lambda_t[G_k] \Lambda_t[G_l]). \end{aligned}$$

Now, since Gell-Mann matrices are traceless one has $\text{Tr}(\Lambda_t[G_k]) = \text{Tr}G_k = 0$. Moreover, using

$$\Lambda_t[G_k] = \sum_m \Delta_{km}(t) G_m,$$

one finds

$$\sum_{k,l} x_k x_l \text{Tr}(\Lambda_t[G_k] \Lambda_t[G_l]) = \sum_{k,l,m} x_k x_l \Delta_{km}(t) \text{Tr}(G_m \Lambda_t[G_l])$$

and using

$$\Delta_{ml}(t) = \text{Tr}(G_m \Lambda_t[G_l])$$

one arrives to

$$\sum_{k,l} x_k x_l \text{Tr}(\Lambda_t[G_k] \Lambda_t[G_l]) = \sum_{k,l,m} x_k x_l \Delta_{km}(t) \Delta_{ml}(t)$$

and finally

$$\|\Lambda_t[X]\|_2^2 = x_0^2 d + \|\Delta_t \mathbf{x}\|_2^2,$$

where now $\|\mathbf{x}\|_2^2 = \sum_k x_k^2$. It is, therefore, clear that

$$\frac{d}{dt}\|\Delta_t \mathbf{x}\|_2 = \frac{d}{dt}\|\Lambda_t[X]\|_2 \leq 0.$$